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# Non-Abelian sine-Gordon solitons

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## Abstract

We point out that non-Abelian sine-Gordon solitons stably exist in the  $U(N)$  chiral Lagrangian. They also exist in a  $U(N)$  gauge theory with two  $N$  by  $N$  complex scalar fields coupled to each other. One non-Abelian sine-Gordon soliton can terminate on one non-Abelian global vortex. They are relevant in chiral Lagrangian of QCD or in color-flavor locked phase of high density QCD, where the anomaly is suppressed at asymptotically high temperature or density, respectively.

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## 1. Introduction

Sine-Gordon kinks (solitons) [1] appear in broad range of physics from classical and quantum field theories [2,3], QCD [4], conformal field theories, integrable systems, and cosmology [5] to condensed matter physics. Condensed matter systems offer a lot of examples of sine-Gordon kinks which can be observed in laboratory experiments, such as Josephson junctions of two superconductors [6], those in multi-layer high  $T_c$  superconductors [7], two-gap superconductors [8–10], chiral p-wave superconductors [11], coherently coupled two-component Bose–Einstein condensates (BECs) [12], two separated BECs with a Josephson coupling [13], helium 3 superfluids [14], and ferromagnets [15]. In particular, sine-Gordon kinks are Josephson vortices in Josephson junctions appearing when a magnetic field is applied parallel to a Josephson junction or layers of high  $T_c$  superconductors [6,7,13,16]. Another interesting case is that a sine-Gordon

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kink connects two fractional vortices winding around different components, to constitute a vortex molecule in multi-gap superconductors [10,17,18] and coherently coupled multi-component BECs [19–21].

Sine-Gordon kinks also explain relations between topological defects or solitons in different dimensions. Sine-Gordon kinks inside the world-volume of a topological defect represent some other topological defects in the bulk; sine-Gordon kinks inside a domain wall are vortices, lumps or baby Skyrmions in the bulk [16,22–24], which explains a relation between sine-Gordon kinks and  $\mathbb{C}P^1$  instantons [25,26]. Sine-Gordon kinks inside a domain wall ring are baby Skyrmions [23]. They represent Skyrmions in the bulk if residing in a domain wall within a domain wall [27–29] or in a vortex string [30,31], they are Hopfions in the bulk if residing in a toroidal domain wall [32], and are Yang–Mills instantons in the bulk if residing inside a monopole string in Yang–Mills theory in  $d = 4 + 1$  dimensions [33].

There have been many proposals of generalizations of the sine-Gordon model. One of such is a complex sine-Gordon model describing a vortex motion in superfluids [34], the  $O(4)$  model [35], conformal field theories [36], and a domain wall junction [37]. There have been non-Abelian generalizations such as the matrix sine-Gordon model [38], the symmetric space sine-Gordon model [39] and so on.

In this paper, we discuss yet another non-Abelian generalization of sine-Gordon kinks. We point out that a sine-Gordon soliton in the  $U(N)$  chiral Lagrangian [40] is a non-Abelian soliton carrying non-Abelian moduli  $\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$ . Here, the term “non-Abelian” is used in the same way with that of non-Abelian vortices [41–44] carrying non-Abelian  $\mathbb{C}P^{N-1}$  moduli, see Refs. [45–47] for a review. As in the same manner with a non-Abelian vortex with non-Abelian moduli which can terminate on a non-Abelian monopole because of the matching of the moduli  $\mathbb{C}P^{N-1}$  [48,49], non-Abelian sine-Gordon kink here can terminate on a non-Abelian global vortex [50–53], see Ref. [4] as a review. We then promote the non-Abelian sine-Gordon solitons to those in non-Abelian  $U(N)$  gauge theories with two  $N$  by  $N$  complex scalar fields coupled to each other by a non-Abelian extension of linear or quadratic Josephson interaction. The Abelian case reduces to phase solitons in two-gap superconductors [8–10], while the non-Abelian extension is relevant to a color superconductor of the color-flavor locking phase of dense QCD matter [4,54].

This paper is organized as follows. In Section 2, after reviewing sine-Gordon kinks in the conventional sine-Gordon model, we discuss non-Abelian sine-Gordon kinks in the  $U(N)$  chiral Lagrangian. In Section 3, sine-Gordon kinks with a modified mass term and their non-Abelian  $U(N)$  generalization are discussed. In Section 4, these sine-Gordon kinks are promoted to gauge theories. The  $U(1)$  gauge theory is nothing but two-gap superconductors or chiral p-wave superconductors corresponding to the conventional or modified mass term, respectively. In Section 5, we discuss that a sine-Gordon kink can terminate on a non-Abelian global vortex. Section 6 is devoted to summary and discussion.

## 2. The sine-Gordon model and chiral Lagrangian

### 2.1. The sine-Gordon model

The sine-Gordon kink is characterized by the first homotopy group  $\pi_1[U(1)] \simeq \mathbb{Z}$ . The Lagrangian density of conventional sine-Gordon model is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \theta)^2 - m^2(1 - \cos \theta) \quad (1)$$

with  $\mu = 0, 1, \dots, d-1$  and  $0 \leq \theta < 2\pi$ . We consider static configurations depending on one spatial direction  $x$ . The static energy density is

$$\mathcal{E} = \frac{1}{2}(\partial_x \theta)^2 + m^2(1 - \cos \theta). \quad (2)$$

The Bogomol'nyi completion for the energy density is obtained as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}(\partial_x \theta)^2 + 2m^2 \sin^2 \frac{\theta}{2} \\ &= \frac{1}{2} \left( \partial_x \theta \mp 2m \sin \frac{\theta}{2} \right)^2 \pm 2m \partial_x \theta \sin \frac{\theta}{2} \\ &\geq \left| 2m \partial_x \theta \sin \frac{\theta}{2} \right| = |t_{\text{SG}}| \end{aligned} \quad (3)$$

with the topological charge density defined by

$$t_{\text{SG}} \equiv 2m \partial_x \theta \sin \frac{\theta}{2} = -4m \partial_x \left( \cos \frac{\theta}{2} \right). \quad (4)$$

The inequality is saturated by the BPS equation

$$\partial_x \theta \mp 2m \sin \frac{\theta}{2} = 0. \quad (5)$$

A single-kink solution interpolating between  $\theta = 0$  at  $x \rightarrow -\infty$  to  $\theta = 2\pi$  at  $x \rightarrow +\infty$  can be given as

$$\theta(x) = 4 \arctan \exp m(x - X) \quad (6)$$

with the position  $X$  in the  $x$ -coordinate. The topological charge for this solution is

$$T_{\text{SG}} = \int dx t_{\text{SG}} = -4m \left[ \cos \frac{\theta}{2} \right]_{x=-\infty}^{x=+\infty} = -4m(-1 - 1) = 8m. \quad (7)$$

The width of the sine-Gordon kink is  $1/m$ .

For later convenience, we introduce a new variable taking a value in the  $U(1)$  group by

$$u \equiv e^{i\theta}. \quad (8)$$

From  $\partial_x \theta = -(i/2)(u^* \partial_x u - (\partial_x u^*)u)$ , the BPS equation is rewritten as

$$-\frac{i}{2}(u^* \partial_x u - (\partial_x u^*)u) \mp m \sqrt{2 - u - u^*} = 0 \quad (9)$$

and the topological charge density is rewritten as

$$t_{U(1)} = -\frac{im}{2}(u^* \partial_x u - (\partial_x u^*)u) \sqrt{2 - u - u^*} = -2m \partial_x \left( \sqrt{2 + u + u^*} \right) \quad (10)$$

The single-kink solution is

$$u(x) = \exp(4i \arctan \exp[m(x - X)]) \quad (11)$$

with the boundary condition  $u \rightarrow 1$  for  $x \rightarrow \pm\infty$ .

## 2.2. Non-Abelian sine-Gordon model as chiral Lagrangian

Here we consider the  $U(N)$  group:

$$U(x) \in U(N) \simeq \frac{U(1) \times SU(N)}{\mathbb{Z}_N} \quad (12)$$

with the first homotopy group is nontrivial:

$$\pi_1[U(N)] = \mathbb{Z}. \quad (13)$$

The Lagrangian for a  $U(N)$  principal chiral model (chiral Lagrangian) for a  $U(N)$ -valued field  $U(x)$  is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \text{tr} \partial_\mu U^\dagger \partial^\mu U - \frac{m^2}{2} \text{tr} (2\mathbf{1}_N - U - U^\dagger) \\ &= \frac{1}{2} \text{tr} (iU^\dagger \partial_\mu U)^2 - \frac{m^2}{2} \text{tr} (2\mathbf{1}_N - U - U^\dagger). \end{aligned} \quad (14)$$

This Lagrangian is invariant under the chiral  $SU(N)_L \times SU(N)_R$  symmetry

$$U(x) \rightarrow V_L U(x) V_R^\dagger, \quad V_{L,R} \in SU(N)_{L,R} \quad (15)$$

The Lagrangian admits the unique vacuum  $U = \mathbf{1}_N$ . The chiral symmetry is spontaneously broken to the vector-like symmetry

$$U(x) \rightarrow V U(x) V^\dagger, \quad V \in SU(N)_{L+R=V}. \quad (16)$$

The energy density for static configuration and its Bogomol'nyi completion are given as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \text{tr} (iU^\dagger \partial_x U)^2 - \frac{m^2}{2} \text{tr} (2\mathbf{1}_N - U - U^\dagger) \\ &= \frac{1}{2} \text{tr} \left[ -\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \mp m \sqrt{2\mathbf{1}_N - U - U^\dagger} \right]^2 \\ &\quad \pm \frac{m}{2} \text{tr} \left[ -\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \sqrt{2\mathbf{1}_N - U - U^\dagger} \right] \\ &\geq |t_{U(N)}|, \end{aligned} \quad (17)$$

with the topological charge, defined by

$$t_{U(N)} \equiv -\frac{m}{2} \text{tr} \left[ i(U^\dagger \partial_x U - \partial_x U^\dagger U) \sqrt{2\mathbf{1}_N - U - U^\dagger} \right]. \quad (18)$$

The BPS equation is obtained as

$$-\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \mp m \sqrt{2\mathbf{1}_N - U - U^\dagger} = \mathbf{0}_N. \quad (19)$$

This equation is invariant under the  $SU(N)$  symmetry in Eq. (16).

Let us construct solutions to this equation. The simplest ansatz is given by the following *Abelian* solution

$$U(x) = u(x) \mathbf{1}_N. \quad (20)$$

By substituting this ansatz into Eq. (19), we find that  $u(x)$  again satisfies Eq. (9). The tension (energy per unit area) of this configuration is  $T = NT_{\text{SG}}$ .

Next, we construct *non-Abelian* solutions. Let us consider the following ansatz [40]:

$$U(x) = \text{diag}(u(x), 1, \dots, 1) \quad (21)$$

By substituting this ansatz into Eq. (19), we find that  $u(x)$  satisfies Eq. (9) and the one-kink solution is obtained as Eq. (11). The tension of this configuration is  $T = T_{SG}$ . Although the solution is obtained by embedding the Abelian solution into the upper-left corner, this solution is truly non-Abelian; in terms of group elements, the ansatz in Eq. (21) can be rewritten as

$$U(x) = \exp\left(i \frac{\theta(x)}{N}\right) \exp(i\theta(x)T_0), \quad (22)$$

$$T_0 \equiv \frac{1}{N} \text{diag}(N-1, -1, \dots, -1).$$

From this expression, one can see that the  $U(1)$  group element rotates only  $2\pi/N$  while the rest is compensated by an  $SU(N)$  group element  $T_0$ . Namely at  $x = \infty$  ( $\theta = 2\pi$ ) the  $U(1)$  group element becomes  $\exp\left(i \frac{2\pi}{N}\right) = \omega$  while the  $SU(N)$  group element becomes  $\exp(2\pi i T_0) = \text{diag}(\omega^{N-1}, \omega^{-1}, \dots, \omega^{-1}) = \omega^{-1} \mathbf{1}_N$ . The  $SU(N)$  group element connects the trivial element to an element of the center  $\mathbb{Z}_N$  of the  $SU(N)$  group.

There is a continuous degeneracy of the solutions with the same energy. Since the Lagrangian and the BPS equation is invariant under the  $SU(N)$  transformation in Eq. (16), the most general solution is obtained as

$$U(x) = V \text{diag}(u(x), 1, \dots, 1) V^\dagger, \quad V \in SU(N). \quad (23)$$

Since there exists a redundancy for the action of  $V$ ,  $V$  in fact takes a value in the coset space

$$V \in \frac{SU(N)}{SU(N-1) \times U(1)} \simeq \mathbb{C}P^{N-1}. \quad (24)$$

Therefore, the one-kink solution has the moduli

$$\mathcal{M} = \mathbb{R} \times \mathbb{C}P^{N-1}. \quad (25)$$

In terms of the group elements, the general solution can be rewritten as

$$U(x) = \exp\left(i \frac{\theta(x)}{N}\right) \exp\left(i\theta(x) V T_0 V^\dagger\right)$$

$$= \exp\left(i \frac{\theta(x)}{N}\right) \exp i \frac{\theta(x)}{N} T, \quad (26)$$

with  $T \equiv V T_0 V^\dagger$ .  $T$  can be any  $SU(N)$  generator normalized as  $e^{i2\pi T} = \omega^{-1} \mathbf{1}_N$ .

Let us introduce the orientational vector  $\phi \in \mathbb{C}^N$  with a constraint

$$\phi^\dagger \phi = 1, \quad (27)$$

which represents homogeneous coordinates of  $\mathbb{C}P^{N-1}$ . The generator  $T$  and the general solution in Eq. (26) can be rewritten by using the orientational vector as

$$T = V T_0 V^\dagger = \phi \phi^\dagger - \frac{1}{N} \mathbf{1}_N, \quad (28)$$

$$U(x) = \exp\left(i\theta(x) \phi \phi^\dagger\right). \quad (29)$$

### 3. The modified sine-Gordon model and chiral Lagrangian

#### 3.1. The modified sine-Gordon model

We consider the Lagrangian density of a sine-Gordon model with an unconventional potential, given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \theta)^2 - m^2 (1 - \cos^2 \theta) \quad (30)$$

with  $\mu = 0, 1$ . This model admits two vacua  $\theta = 0, \pi$  in the defined range  $0 \leq \theta \leq 2\pi$ . We concentrate on static configurations. The static energy density is

$$\mathcal{E} = \frac{1}{2}(\partial_x \theta)^2 + m^2 (1 - \cos^2 \theta) = \frac{1}{2}(\partial_x \theta)^2 + m^2 \sin^2 \theta. \quad (31)$$

The Bogomol'nyi completion for the energy density is obtained as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \left[ (\partial_x \theta)^2 + 2m^2 \sin^2 \theta \right] \\ &= \frac{1}{2} \left( \partial_x \theta \mp \sqrt{2}m \sin \theta \right)^2 \pm \sqrt{2}m \partial_x \theta \sin \theta \\ &\geq \left| \sqrt{2}m \partial_x \theta \sin \theta \right| = |t_{\text{SG}}| \end{aligned} \quad (32)$$

with the topological charge density

$$t_{\text{SG}} \equiv \sqrt{2}m \partial_x \theta \sin \theta = -\sqrt{2}m \partial_x (\cos \theta). \quad (33)$$

The inequality is saturated by the BPS equation

$$\partial_x \theta \mp \sqrt{2}m \sin \theta = 0. \quad (34)$$

A one-kink solution interpolating between  $\theta = 0$  at  $x \rightarrow -\infty$  to  $\theta = \pi$  at  $x \rightarrow +\infty$  can be given as

$$\theta(x) = 2 \arctan \exp \sqrt{2}m(x - X) \quad (35)$$

with the position  $X$  in the  $x$ -coordinate and the width  $1/m$ . The topological charge for this solution is

$$T_{\text{SG}} = \int dx t_{\text{SG}} = -\sqrt{2}m [\cos \theta]_{x=-\infty}^{x=+\infty} = -\sqrt{2}m(-1 - 1) = 2\sqrt{2}m. \quad (36)$$

In terms of  $u(x) = e^{i\theta(x)}$ , the BPS equation is rewritten as

$$\begin{aligned} \partial_x u \mp \frac{\sqrt{2}}{2}m(1 - u^2) &= 0, \\ (\Leftrightarrow -i(u^* \partial_x u - (\partial_x u^*)u) \mp \sqrt{2}im(u - u^*) &= 0), \end{aligned} \quad (37)$$

and the topological charge density is rewritten as

$$t_{\text{U}(1)} = -\frac{\sqrt{2}}{2}mu^* \partial_x u(u - u^*) \quad (38)$$

The one-kink solution is

$$u(x) = \exp \left( 2i \arctan \exp \frac{\sqrt{2}m}{4}(x - X) \right). \quad (39)$$

### 3.2. Non-Abelian sine-Gordon model as chiral Lagrangian with modified mass

The Lagrangian for  $U(N)$  principal chiral model with a modified mass is

$$\mathcal{L} = \frac{1}{2} \text{tr} \partial_\mu U^\dagger \partial^\mu U - V = \frac{1}{2} \text{tr} (i U^\dagger \partial_\mu U)^2 - V \quad (40)$$

$$\begin{aligned} V &= m^2 \text{tr} (2\mathbf{1}_N - U^2 - U^{\dagger 2}) = m^2 \text{tr} (2\mathbf{1}_N - U - U^\dagger)(2\mathbf{1}_N + U + U^\dagger) \\ &= m^2 \text{tr} (\mathbf{1}_N - U^2)(\mathbf{1}_N - U^{\dagger 2}). \end{aligned} \quad (41)$$

This model admits two vacua  $U = \pm \mathbf{1}_N$ . The energy density for static configuration and its Bogomol'nyi completion are given as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \text{tr} \partial_x U^\dagger \partial_x U + m^2 \text{tr} (\mathbf{1}_N - U^2)(\mathbf{1}_N - U^{\dagger 2}) \\ &= \frac{1}{2} \text{tr} \left[ \{ \partial_x U^\dagger \mp \sqrt{2}m(\mathbf{1}_N - U^{\dagger 2}) \} \{ \partial_x U \mp \sqrt{2}m(\mathbf{1}_N - U^2) \} \right] \\ &\quad \pm 2m \text{tr} \left[ \partial_x U^\dagger (\mathbf{1}_N - U^2) + \partial_x U (\mathbf{1}_N - U^{\dagger 2}) \right] \\ &\geq |t_{U(N)}|, \end{aligned} \quad (42)$$

with the topological charge, defined by

$$\begin{aligned} t_{U(N)} &\equiv \sqrt{2}m \text{tr} \left[ \partial_x U^\dagger (\mathbf{1}_N - U^2) + \partial_x U (\mathbf{1}_N - U^{\dagger 2}) \right] \\ &= \sqrt{2}m \text{tr} \left[ U^\dagger \partial_x U (U - U^\dagger) + \text{h.c.} \right]. \end{aligned} \quad (43)$$

The BPS equation is obtained as

$$\begin{aligned} \partial_x U \mp \sqrt{2}m(\mathbf{1}_N - U^2) &= \mathbf{0}_N \\ \Leftrightarrow (iU^\dagger \partial_x U \mp \sqrt{2}im(U^\dagger - U)) &= \mathbf{0}_N. \end{aligned} \quad (44)$$

As in the same manner, the Abelian kink in Eq. (39) can be embedded into a conner as in Eq. (21) to obtain a non-Abelian kink. Also, it allows the  $\mathbb{C}P^{N-1}$  moduli as Eq. (23).

## 4. Non-Abelian sine-Gordon soliton in gauge theories

### 4.1. Abelian gauge theory: two-gap superconductors and chiral p-wave superconductors

Let us consider a  $U(1)$  gauge theory coupled with two complex scalar fields  $\phi_i(x)$  ( $i = 1, 2$ ), given by

$$\mathcal{L} = \frac{1}{2} \sum_{i=1,2} D_\mu \phi_i^* D^\mu \phi_i + \mathcal{L}_J - \sum_{i=1,2} \frac{\lambda_i}{4} (|\phi_i|^2 - 1)^2 + \frac{1}{4e^2} F_{\mu\nu}^2 \quad (45)$$

with  $D_\mu \phi_i = (\partial_\mu - iA_\mu)\phi_i$ .  $\mathcal{L}_J$  is a Josephson term either linear or quadratic:

$$\begin{aligned} \mathcal{L}_{J,1} &= \frac{\gamma}{2} (\phi_1^* \phi_2 + \text{c.c.} - 2) \\ \mathcal{L}_{J,2} &= \frac{\gamma}{2} [(\phi_1^* \phi_2)^2 + \text{c.c.} - 2]. \end{aligned} \quad (46)$$

The gauge transformation is defined by

$$\phi_i \rightarrow e^{i\alpha(x)} \phi_i, \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x), \quad (47)$$

while a  $U(1)$  global transformation

$$\phi_1 \rightarrow e^{i\beta} \phi_1, \quad \phi_2 \rightarrow e^{-i\beta} \phi_2 \quad (48)$$

is explicitly broken by  $\gamma \neq 0$ .

Let us take strong coupling limit (with keeping  $\gamma$  finite):

$$e, \lambda_i \rightarrow \infty, \quad (49)$$

giving constraints

$$|\phi_i| = 1, \quad \phi_i = e^{i\theta_i}. \quad (50)$$

With taking a gauge  $A_\mu = \partial_\mu \theta_2$  and defining the phase difference  $\theta(x) \equiv \theta_1(x) - \theta_2(x)$ , the covariant derivative terms in Lagrangian in Eq. (45) become

$$\begin{aligned} D_\mu \phi_1 &= i(\partial_\mu \theta_1 - A_\mu) e^{i\theta_1} = i\partial_\mu (\theta_1 - \theta_2) e^{i\theta_1} = i\partial_\mu \theta e^{i\theta_1}, \\ D_\mu \phi_2 &= i(\partial_\mu \theta_2 - A_\mu) e^{i\theta_2} = 0, \end{aligned} \quad (51)$$

while the Josephson terms in Eq. (46) become

$$\mathcal{L}_{J,1} = -m^2(1 - \cos \theta), \quad \mathcal{L}_{J,2} = -m^2(1 - \cos^2 \theta), \quad \gamma \equiv m^2. \quad (52)$$

The gauge theory Lagrangian in Eq. (45) reduces the sine-Gordon model in Eq. (1) or the modified sine-Gordon model in Eq. (30).

Let us remark on physical realizations of this model and its sine-Gordon solitons. A non-relativistic version of the Lagrangian has the kinetic and gradient terms

$$\frac{1}{2} \sum_i (i\phi_i^* D_0 \phi_i + \text{h.c.} - D_a \phi_i^* D_a \phi_i), \quad (53)$$

instead of the first term in the Lagrangian in Eq. (45). Here  $a = 1, 2, (3)$  is a spatial index. The linear Josephson term  $\mathcal{L}_{J,1}$  in Eq. (46) is relevant for the Landau–Ginzburg description of two-gap superconductors such as  $\text{MgB}_2$ , in which the term proportional to  $\gamma$  is called the (internal) Josephson coupling and  $\theta(x)$  is called the Leggett mode. The sine-Gordon soliton is called the phase soliton in this context, which was first pointed out theoretically [8] and was found experimentally [9]. It is also relevant for a Josephson junction of two superconductors. On the other hand, the case with the quadratic Josephson interaction  $\mathcal{L}_{J,2}$  in Eq. (46) is relevant for chiral p-wave superconductors [11], such as  $\text{Sr}_2\text{RuO}_4$ .

A non-relativistic version of the Lagrangian (53) in which overall  $U(1)$  is not gauged ( $e = 0$ ) yields the Gross–Pitaevskii equation for two-component Bose–Einstein condensates of ultracold atomic gases such as  $\text{Rb}_{87}$ , in which the term proportional to  $\gamma$  is called a Rabi oscillation term. (In addition, the term  $g_{12}|\phi_1|^2|\phi_2|^2$  is also present but it is not important for the phase solitons.) The sine-Gordon (phase) soliton in this case was studied in Ref. [12].



#### 4.2. Non-Abelian gauge theory

Let us consider a  $U(N)$  gauge theory coupled with two  $N \times N$  matrix-valued complex scalar fields  $\Phi_i(x)$  ( $i = 1, 2$ ), whose Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \sum_{i=1,2} \text{tr} D_\mu \Phi_i^\dagger D^\mu \Phi_i + \frac{\gamma}{2} \text{tr} (\Phi_1^\dagger \Phi_2 + \text{h.c.} - 2\mathbf{1}_N) \\ & - \sum_{i=1,2} \frac{\lambda_i}{4} \text{tr} (\Phi_i^\dagger \Phi_i - \mathbf{1}_N)^2 + \frac{1}{4g^2} \text{tr} F_{\mu\nu}^2 \end{aligned} \quad (54)$$

with  $D_\mu \Phi_i = (\partial_\mu - iA_\mu) \Phi_i$  and  $A_\mu = A_\mu^A(x) T_A$  with  $U(N)$  generators  $T_A$ .  $\mathcal{L}_J$  is a non-Abelian Josephson term either linear or quadratic:

$$\begin{aligned} \mathcal{L}_{J,1} &= \frac{\gamma}{2} \text{tr} (\Phi_1^\dagger \Phi_2 + \text{h.c.} - 2\mathbf{1}_N), \\ \mathcal{L}_{J,2} &= \frac{\gamma}{2} \text{tr} [(\Phi_1^\dagger \Phi_2)^2 + \text{h.c.} - 2\mathbf{1}_N]. \end{aligned} \quad (55)$$

The  $U(N)_V$  gauge transformation is defined by

$$\Phi_i \rightarrow V(x) \Phi_i, \quad A_\mu \rightarrow V(x) A_\mu V(x)^{-1} + iV(x) \partial_\mu V^{-1}(x), \quad (56)$$

while a  $U(N)_A$  global transformation

$$\Phi_1 \rightarrow g \Phi_1, \quad \Phi_2 \rightarrow g^{-1} \Phi_2, \quad g \in U(N)_A \quad (57)$$

is explicitly broken by  $\gamma \neq 0$ .

Let us take strong coupling limit (with keeping  $\gamma$  finite):

$$g, \lambda_i \rightarrow \infty, \quad (58)$$

giving constraints

$$\Phi_i^\dagger \Phi_i = \mathbf{1}_N. \quad (59)$$

These constraints can be solved as

$$\Phi_1(x) = \hat{U}(x), \quad \Phi_2(x) = \hat{U}^\dagger(x), \quad \hat{U}(x) \in U(N). \quad (60)$$

With taking a gauge  $A_\mu = i\hat{U}^\dagger \partial_\mu \hat{U}$  and defining  $U(x) \equiv \hat{U}^2(x)$ , the covariant derivative terms in Lagrangian in Eq. (54) become

$$D_\mu \Phi_1 = \partial_\mu \hat{U} - iA_\mu \hat{U} = \partial_\mu U(x), \quad D_\mu \Phi_2 = \partial_\mu \hat{U}^\dagger - iA_\mu \hat{U}^\dagger = 0, \quad (61)$$

and the Josephson terms reduce to

$$\begin{aligned} \mathcal{L}_{J,1} &= -m^2 \text{tr} (2\mathbf{1}_N - U - U^\dagger), \\ \mathcal{L}_{J,2} &= -m^2 \text{tr} (2\mathbf{1}_N - U^2 - U^{\dagger 2}), \\ \gamma &\equiv m^2. \end{aligned} \quad (62)$$

Therefore, the gauge theory Lagrangian in Eq. (54) reduces the non-Abelian sine-Gordon model in Eq. (14) or the modified non-Abelian sine-Gordon model in Eq. (30).

The relativistic Lagrangian in Eq. (54) is relevant for a linear model description of chiral Lagrangian using a hidden local gauge symmetry for which gauge bosons of  $U(N)$  gauge symmetry is vector mesons of the hidden local symmetry, see, e.g. Ref. [55].

A non-relativistic version of the Lagrangian has the kinetic and gradient terms

$$\frac{1}{2} \sum_i \text{tr} (i \Phi_i^\dagger D_0 \Phi_i + \text{h.c.} - D_a \Phi_i^* D_a \Phi_i) \quad (63)$$

instead of the first term in the Lagrangian in Eq. (54). The non-relativistic case with  $N = 3$  with ungauged  $U(1)$  is relevant for the Landau–Ginzburg description of the color-flavor locking phase (a color superconductor) for high density QCD [4,54]. In this case,  $(\Phi_1)_{\alpha i} = \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} q_{j\beta}^L q_{k\gamma}^L$  and  $(\Phi_2)_{\alpha i} = \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} q_{j\beta}^R q_{k\gamma}^R$  are diquark condensates of left and right handed quarks  $q_{j\beta}^L$  and  $q_{j\beta}^R$ , respectively, where  $\alpha, \beta, \gamma = 1, 2, 3$  and  $i, j, k = 1, 2, 3$  are color and flavor indices, respectively.

Here, we have considered the potential for the  $U(1)$  symmetry induced from quark mass in chiral Lagrangian in QCD. On the other hand, there is another potential term  $V \sim \det \Phi_1 + \det \Phi_2$  induced from the  $U(1)_A$  anomaly at quantum level. The non-Abelian sine-Gordon kink should be deformed by this potential accordingly [56]. Therefore, in real QCD, our solutions are relevant in asymptotically high temperature or high density, in which the  $U(1)_A$  anomaly disappears.

## 5. Non-Abelian vortex that terminates non-Abelian sine-Gordon kink

The  $U(N)$  chiral Lagrangian or more precisely the corresponding  $U(N)$  linear sigma model admits a non-Abelian global vortex [50–53], see Ref. [4] as a review. When one discusses the asymptotic form of the vortex solution, the chiral Lagrangian is enough. Here, we briefly discuss a relation between the non-Abelian global vortex and the non-Abelian sine-Gordon kink.

Let  $(r, \varphi, z)$  be cylindrical coordinates of space. Then, the asymptotic form of a non-Abelian global vortex can be written as

$$U(r \rightarrow \infty, \varphi, z) = \text{diag}(e^{i\theta(\varphi)}, 1, \dots, 1). \quad (64)$$

In the limit of no mass term ( $m = 0$ ), the unit winding solution is simply given by  $\theta = \varphi$  so that the vortex is axisymmetric. The configuration in Eq. (64) can be rewritten as

$$U(r \rightarrow \infty, \varphi, z) = \exp \left( i \frac{\theta(\varphi)}{N} \right) \exp (i\theta(x) T_0), \quad (65)$$

$$T_0 \equiv \frac{1}{N} \text{diag}(N-1, -1, \dots, -1).$$

It is obvious that the configuration of the vortex breaks the  $SU(N)_V$  symmetry of the vacuum to a subgroup  $SU(N-1) \times U(1)$  so that there appear moduli  $\mathbb{C}P^{N-1}$ , although these moduli are non-normalizable [51,52].

In the presence of the mass term ( $m \neq 0$ ), the global vortex configuration is deformed and is no more axisymmetric. In this case, the potential term appears for the field  $\theta(\varphi)$  in the vortex ansatz in Eq. (64). This is of course the sine-Gordon potential discussed in the previous sections. Only the difference is the argument of  $\theta$  is  $\theta(\varphi)$  here and  $\theta(x)$  before. The final configuration is a non-Abelian vortex attached by a non-Abelian sine-Gordon kink, as schematically drawn in

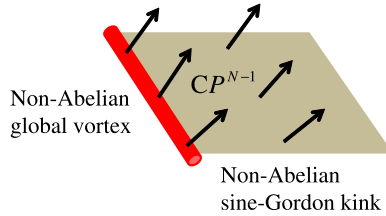


Fig. 1. A junction of a non-Abelian vortex and a non-Abelian sine-Gordon domain wall. A non-Abelian vortex is attached by a non-Abelian sine-Gordon kink in the presence of the mass. In other words, the latter can terminate on the former. The  $\mathbb{C}P^{N-1}$  moduli, that are denoted by arrows, match at the junction line.

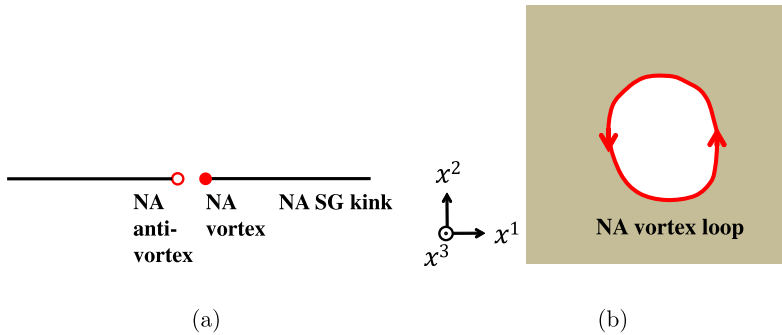


Fig. 2. Decay of a non-Abelian sine-Gordon kink. (a) In  $d = 2 + 1$ , a non-Abelian sine-Gordon domain line can decay by creating a pair of a non-Abelian vortex and a non-Abelian anti-vortex. (b) In  $d = 3 + 1$  the non-Abelian sine-Gordon domain wall can decay by creating a hole bound by a closed non-Abelian vortex string. These processes can occur either thermally or by quantum tunneling.

Fig. 1. Both the non-Abelian vortex and non-Abelian sine-Gordon kink have the  $\mathbb{C}P^{N-1}$  moduli, and consequently they match at a junction line.<sup>1</sup>

This fact implies the instability of sine-Gordon kinks in the  $U(N)$  linear sigma model as in the same manner with an axion string [5]. In  $d = 2 + 1$ , the sine-Gordon domain line can terminate on a global non-Abelian vortex [4,50–53]. The domain line can decay by creating a pair of a non-Abelian vortex and a non-Abelian anti-vortex, as shown in Fig. 2(a). In  $d = 3 + 1$ , the non-Abelian sine-Gordon domain wall can decay by creating a hole bound by a closed non-Abelian vortex string, as illustrated in Fig. 2(b). This process can occur either thermally or by quantum tunneling. More details will be discussed elsewhere. However, note that the instability does not exist in the nonlinear model, the  $U(N)$  chiral Lagrangian. This is the same situation with an axion string [5].

## 6. Summary and discussion

We have pointed out that the  $U(N)$  chiral Lagrangian admits a non-Abelian sine-Gordon kink that carries non-Abelian moduli  $\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$ . We have also presented

<sup>1</sup> This situation is the same with the case of a non-Abelian monopole that terminates a non-Abelian vortex string [48, 49]. This not only because the both  $\mathbb{C}P^{N-1}$  moduli of the monopole and vortex match at junction point but also because the monopole moduli are non-normalizable.

the non-Abelian gauge theory that admits the same non-Abelian sine-Gordon kink. In the Abelian case, this reduces to the Lagrangian for two-gap superconductors. Two possibilities to realize it in QCD have been discussed. We have also briefly discussed in the  $U(N)$  linear sigma model that a sine-Gordon kink can terminate on a non-Abelian global vortex, implying the instability of the sine-Gordon kink in the linear model.

Several discussions are addressed here. One of the most important task remaining is constructing the low-energy effective theory by the moduli approximation [57], which is the  $\mathbb{C}P^{N-1}$  model. One then can construct  $\mathbb{C}P^{N-1}$  lumps on it that would represent  $U(N)$  Skyrmions as was so for the  $SU(2)$  model with two vacua [27,28]. See Ref. [58] for a further study along this line.

The interaction between two kinks located at  $x = X_{1,2}$  with the orientations  $\phi_{1,2}$  can be considered. Like the Abrikosov-type ansatz for vortices, we can give an ansatz for the total configuration as  $U_{\text{tot}}(x) = U_1(x - X_1, \phi_1)U_2(x - X_2, \phi_2)$  for well-separated kinks  $|X_1 - X_2| \gg m^{-1}$ . In particular, an Abelian sine-Gordon kink would be separated into  $N$  non-Abelian kinks without cost of energy, which can be expected from the fact that an Abelian kink has energy  $N$  multiple of those of non-Abelian kinks. A similar calculation was done for the force between two non-Abelian global vortices [4,52].

In two-gap superconductors, a unit winding vortex can be split into two fractional vortices winding around different components, which are connected by a sine-Gordon kink [10,17,18]. The same happens for coherently coupled multi-component BECs [19–21]. In the same way, a local non-Abelian vortex can be split into a set of two global non-Abelian vortices connected by a non-Abelian sine-Gordon domain wall discussed here. In the case of the color-flavor locked phase of dense quark matter, a non-Abelian vortex [4,59] has  $1/3$  fractional  $U(1)$  winding in both  $\Phi_1$  and  $\Phi_2$ , but it may be decomposed into a global vortex with  $1/6$   $U(1)$  winding ( $1/3$   $U(1)$  winding in only one of  $\Phi_1$  and  $\Phi_2$ ). This will be also discussed elsewhere.

The  $U(N)$  principal chiral model studied in this paper has been found to appear as the effective theory of a non-Abelian domain wall [60]. If a Josephson term is added in the bulk theory, this domain wall behaves as a Josephson junction of two color superconductors and the mass term is induced in the  $U(N)$  principal chiral model on the wall [61]. Then, non-Abelian sine-Gordon solitons describe non-Abelian Josephson vortices, that is, non-Abelian vortices trapped inside the Josephson junction [61,62].

Non-Abelian  $U(N)$  sine-Gordon kinks can be extended to the case of arbitrary gauge groups  $G$  in the form of  $\frac{G \times U(1)}{\mathbb{Z}_r}$  with the center  $\mathbb{Z}_r$  of  $G$ , since non-Abelian vortices with this type of gauge groups were studied before [63], such as  $SO(N)$  and  $USp(2N)$  groups [64].

Finally, the sine-Gordon model is integrable. Therefore, we expect the non-Abelian sine-Gordon model presented here is also integrable.

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